Analysis of a Wheel-Damped Reaction Boom Control System

Eugene H. Thompson* and Arthur G. Buckingham†
Westinghouse Defense and Space Center, Baltimore, Md.

A conceptually simple and accurate small deformation model is developed herein for a relatively new type of attitude control system. This hybrid system utilizes a pivoted, flexible boom with tip mass as the actuator and small inertia wheels to dampen the gravity induced boom librations. The representation of boom flexure by N normal modes provides a well established basis for treating the shear discontinuity produced by the boom tip mass, the dynamic constraint imposed by the applications of a controlled torque at the boom's base, and the controlled momentum exchange between the satellite and the inertia wheel. The resultant model requires a minimum of coordinates and a relatively simple spring matrix at the expense of a heavily encumbered mass matrix. Analog simulation results show that the response of the pivoted boom to a torque applied at its base exhibits both resonances and antiresonances. Thus, the effective phase shift is not accumulative. The resonances occur at those frequencies characterizing a classic pin-free boom with tip mass, whereas the antiresonances occur near, but somewhat higher, than the characteristic frequencies of a classical, cantilevered boom with tip mass. This upward shift in the antiresonant frequencies results from the boom's attachment to a body having finite inertia.

Nomenclature

$\mathbf{b}$ = position of a point on the elastic boom relative to the pivot
D = boom diameter
D_B = torque-angular rate gain of the boom control torquer
D_w = torque-angular rate gain of the wheel control torquer
EI = bending rigidity of the uniform boom
f_i = pin-free mode shapes of the uniform boom
$\mathbf{F}_J,F_J$ = station-keeping thrust vector and its magnitude
$\mathbf{g},g$ = position of the gimbal joint relative to the main vehicle's mass center, and its magnitude
I_F = principal mass moment of inertia of the main vehicle
I_W = principal mass moment of inertia of the reaction wheel
I_U = mass moment of inertia of the rigid, uniform boom about the pivot
k = modal constant
K_B = torque-angular position gain of the boom control torquer
K_W = torque-angular position gain of the wheel control torquer
L = boom length
l = distance from the pivot to a point on the boom's undeformed centerline
M,dm = mass; differential mass
$\mathbf{P}$ = position of any differential mass relative to the Earth's center
Q = a generalized force
$q,\delta q$ = a generalized coordinate; a virtual displacement
$\mathbf{R}$ = position of the composite mass center relative to the main vehicle's center of mass
$\mathbf{r}$ = position of a differential mass in the main vehicle relative to the main vehicle's center of mass
T = absolute kinetic energy of any system member
$[\mathbf{T}]$ = orthogonal transformation from the reference triad to an inertial triad
$\mathbf{v}$ = unitary local vertical vector
V_S = strain energy in bending of the uniform boom
w = total bending deflection of the boom at any point relative to its undeformed centerline
W = virtual work
W_i = contribution of a particular mode to boom tip flexure (modal amplitude)
$(\beta_i L)$ = eigenvalues for pin-free modes of the uniform boom
β = angular orientation of the boom's undeformed centerline relative to the main vehicle
δ = total bending deflection at the boom tip relative to its undeformed centerline
ϵ = gimbal angle
θ = angular orientation of the main vehicle with respect to the local vertical
θ_i = commanded orientation of the main vehicle
ρ_u = uniform boom mass per unit length
T_B = boom gimbal control torque
T_W = reaction wheel control torque
ω_o = orbital angular rate
ω_w = reaction wheel speed relative to the main vehicle

Subscripts

B = boom with tip mass
C,S,T = cosine, sine, and tangent
$2C,2S$ = cosine, and sine of twice the angle
f_i = a flexure mode of the composite boom
F = main vehicle
J = station-keeping jet
T = boom tip mass
U = uniform boom
W = reaction wheel

Operators

$\partial()/\partial()$ = partial derivative
$(\ )^2$ = dot product of a vector with itself
$d(\ )/dt$ = total time derivative
$\cdot$ = time derivative of the vector magnitude
$[\ ]^T$ = transpose of a matrix
$'$ = derivative with respect to length

Received November 19, 1968; revision received September 8, 1969. This research was supported by NASA Goddard Space Flight Center, contract NAS-5-10257 (Amendment 2). Project Director—J. A. Gatlin.

* Senior Engineer, Integrated Avionics. Associate Member AIAA.
† Advisory Engineer, Integrated Avionics. Associate Fellow AIAA.

Introduction

Because of their low pointing accuracy (2–5°) and limited response time, gravity-gradient satellites have generally been employed in missions where reliability is a key factor and attitude control requirements are modest. Although many long life missions are foreseeable, few of them are likely to accept pointing errors as large as several degrees. Consequently, there has been a recent trend toward hybrid attitude control systems that merge gravity-gradient and reaction-control techniques to obtain better pointing accuracies and, hopefully, maintain the reliability of the passive systems.

This paper deals with a relatively new member of this class of attitude control systems which have received little attention in the current literature. The system employs an extendible boom with tip mass that is attached to the satellite through a torque-controlled, two-axis gimbal. Two-axis position control is obtained by using the pivoted boom with tip mass as a reaction element, and a horizon sensor to establish the vertical. Gimbal angular rate provides sufficient damping of the satellite attitude motion. Two small inertia wheels dampen the gravity-induced librations of the boom and a third provides conventional wheel control about the remaining axis, when required. The pivoted boom enables continuous momentum dumping from all three inertia wheels.

The basic feasibility of this control concept has been established in Ref. 1 in which the flexible boom is treated as a single rigid segment with a spring at its base. Because the explicit form and capabilities of this control system are strongly influenced by the flexural behavior of the boom, a more comprehensive dynamic model is needed. Recently, several papers have been devoted to the general problem of dynamically modelling large flexible bodies and rigid bodies with flexible appendages. Graham[2] and Dow[3] have analyzed specific configurations utilizing in their basic model rigid rods connected by artificial springs with lumped masses at the joints. In more general treatments, Hooker and Margulies,[4] and Roberson and Wittenburg[5] have developed dyadic equations describing the motion of an arbitrary number N of interconnected rigid bodies. Flexible bodies, including the pivoted boom of interest here, may be treated by choosing a large N (e.g., two or three segments are often needed for each mode). As noted by the above authors, computational efficiency rapidly limits the practical choice of N. Ashley,[6] Liu and Mitchell,[7] and Farrell[8] have treated large flexible bodies and rigid bodies with flexible appendages utilizing normal mode techniques. Although this approach reduces problem dimensions to a minimum, these papers are generally concerned with passive configurations characterized by free-free shapes or cantilever attachments, and in most cases are restricted to a single mode. The present paper provides a modest extension to the case wherein the boom is allowed N modes of bending flexure with a significant shear discontinuity at its tip; the attachment is pivotal and is constrained in a controlled manner, and the satellite exchanges momentum in a controlled manner with a discrete reaction element. The model is restricted to a single plane in which the boom undergoes pure bending. An interlocked seam is assumed for the boom; thus, the higher frequency longitudinal and torsional vibrations may be safely ignored. Although a planar analysis is not conclusive, it provides valuable insight into the influence of the pivoted boom's dynamic behavior on the design of a reaction boom control system.

Generalized Variables

The generalized coordinates needed to characterize the degrees of freedom of the composite satellite are depicted in Fig. 1. Angular orientation of the main vehicle (control-antenna-feed structure) is conveniently referenced to the local vertical by the coordinate θ. There are many possible choices to represent the elastic degrees of freedom of the gravity-gradient boom. For example, either discrete or distributed coordinates may be used.[9] Further, these coordinates might be associated with such modal characterizations as cantilever or pin-free. In this paper, the distributed coordinates or modal amplitudes, W_i, associated with pin-free modes of the uniform boom yield a suitable deflection

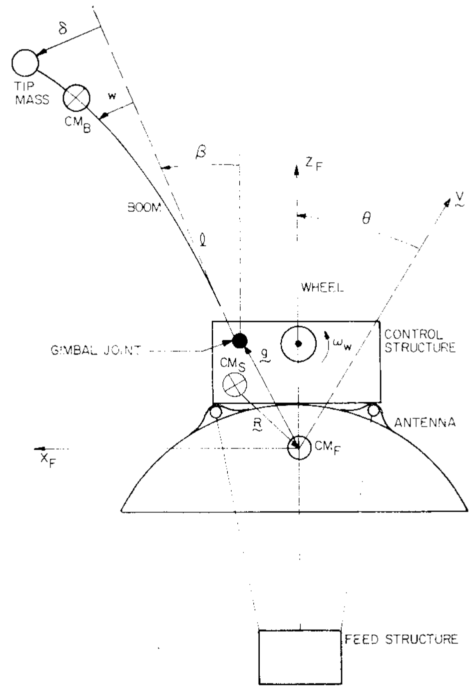

Fig. 1 Lagrangian coordinates.

profile. By normalizing these mode shapes to unity at the boom tip, the W_i also represent boom tip deflections relative to its undeformed centerline. The pivotal degree of freedom of the boom (zero frequency pin-free mode) or the angular orientation of its undeformed centerline relative to the main vehicle is denoted by the coordinate β. Finally, ω_w denotes the reaction wheel speed relative to the main vehicle, its angular orientation being irrelevant.

Because the time periods of interest here are short, the orbital motion of the composite mass center is assumed to be circular and unperturbed. Thus, the above coordinates completely characterize the system's allowable degrees of freedom. Relative motion between the mass centers of the various structural members is implicit in the absolute kinetic energy formulated in terms of these coordinates.

Kinetic Energy of the Composite Satellite

Generally, the total kinetic energy of an elastic structure is formulated by integrating the kinetic energy of a differential mass over the entire structure. To account for the inertia wheel in this system, the above formulation must be augmented by the rotational energy associated with this discrete reaction element. The absolute kinetic energy of the system, then, is

$$T_s = \tfrac{1}{2}\int_s (d\mathbf{P}/dt)^2 dm_s + T_w \qquad (1)$$

where the subscript s refers to the composite satellite.

Since kinetic energy is a scalar, it is independent of the manner in which the structural member velocities are coordinatized. By choosing the body-fixed triad $(X_F - Y_F - Z_F)$ shown in Fig. 1, whose origin is at the main vehicle's mass center, the system kinetic energy is well defined in terms of the previously chosen generalized coordinates. Recognizing that the kinetic energy contribution from the discrete tip mass is $M_T/2[d(\mathbf{R} + \mathbf{g} + \mathbf{b}(L))/dt]^2$ the techniques presented in Ref. 8 enable Eq. (1) to be written explicitly in terms of the position vectors of the composite mass center ($\mathbf{R}$) and an arbitrary point on the deformed boom ($\mathbf{b} + \mathbf{g}$). So

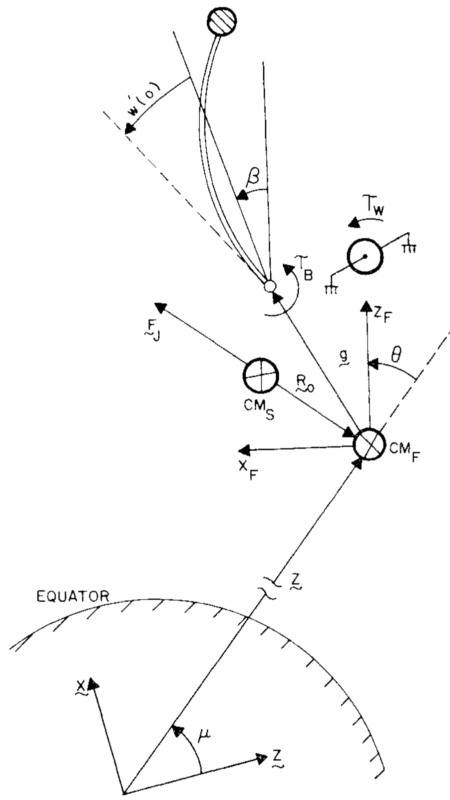

Fig. 2 Generalized forces.

$$
\begin{aligned}
T_s = &-\frac{1}{2}M_s\left(\frac{d\mathbf{R}}{dt}\right)^2 - M_s\frac{d\mathbf{g}^T}{dt}\frac{d\mathbf{R}}{dt} - \frac{1}{2}M_B\left(\frac{d\mathbf{g}}{dt}\right)^2 + \\
&\frac{1}{2}M_T\left(\frac{d\mathbf{b}(L)}{dt}\right)^2 + \frac{1}{2}I_F(\dot\theta + \omega_0)^2 + \\
&\frac{1}{2}I_w(\omega_w + \dot\theta + \omega_0)^2 + \frac{1}{2}I_u(\dot\beta + \dot\theta + \omega_0)^2 + \\
&\frac{M_u}{8}\left[(\dot\beta + \dot\theta + \omega_0)^2 \sum_i W_i^2 + \Sigma(\dot W_i)^2\right] + \\
&(\dot\beta + \dot\theta + \omega_0)L\sum_i \dot W_i m_i
\end{aligned} \tag{2}
$$

where $m_i = k_iM_u/(\beta_iL)$. Since $\mathbf{g}$ is independent of time, $\mathbf{R}$ and $\mathbf{b}$ will be expressed in terms of the previously chosen generalized coordinates in order to obtain T_s in the form required by a Lagrangian analysis. The position of any point on the deformed boom relative to the gimbal joint can be written by inspection (Fig. 1) as

$$
\mathbf{b} = \begin{bmatrix} l\beta_s + w\beta_c \\ 0 \\ l\beta_c - w\beta_s \end{bmatrix} \tag{3}
$$

where $w = \Sigma\, W_if_i$ for small deformations. Taking the first mass moment about the origin of the reference triad, $\mathbf{R}$ is found to be

$$
\mathbf{R} = \begin{bmatrix} M_1g_1 + M_2L\beta_s + M_3\delta\beta_c \\ 0 \\ M_1g_2 + M_2L\beta_c + M_3\delta\beta_s \end{bmatrix} \tag{4}
$$

where $M_1 = M_B/M_s$, $M_2 = (M_T + M_u/2)/M_s$, $M_3 = M_T/M_s$, and $M_4 = M_u/M_sL$. The desired T_s is obtained by substituting Eqs. (3) and (4) into Eq. (2).

Potential Energy of the Composite Satellite

Because the bending deflections and time periods treated here are both small, boom flexure is considered to be conservative and its stress-strain characteristic to be linear. Reference 10 gives the strain energy of this ideal elastic structure as

$$
V_S = \frac{EI}{2}\int_0^L (w'')^2 dl
$$

The flexural rigidity of the boom is assumed to be uniform and l need not be corrected for arc length. Utilizing the classical stiffness coefficients in Ref. 11, the strain energy reduces to $V_S = (EIL/8)\,\Sigma\, W_i^2\beta_i^4$. Gravity-gradient induced disturbances are adequately treated as constants over the frequency range of interest here.

Generalized Forces

Boom and inertia wheel control torques, and the orbital station-keeping thrust shown in Fig. 2 constitute the significant generalized forces not derivable from a potential function. Since the earlier kinetic energy formulation accounts for all D'Alembert-type forces and torques, the internal reaction torques applied by the boom and inertia wheel torquers to the main vehicle need not be included in the generalized forces.

Assuming independent virtual displacements in the generalized coordinates, the infinitesimal virtual work done by the boom torquer is

$$
\delta W_B \triangleq \sum_n Q_{Bn}\delta q_n = Q_{BB}\delta\beta + \sum_i Q_{Bf_i}\delta W_i \tag{5}
$$

where $Q_{B\theta}$ need not be included. Also, the boom torquer does no work on a virtual displacement in the inertia wheel's orientation. Since the pin-free boom has a time-varying slope at its base (see Fig. 2), δW_B is also given by classical mechanics as $\delta W_B = T_B[\delta\beta + \delta w'(0)]$. Equating like terms with Eq. (5), the generalized forces applied by the boom torquer are $Q_{BB} = T_B$ and $Q_{Bf_i} = T_Bf'_i(0)$. Previous work has shown that the boom torquer is adequately represented by $T_B = -K_B(\theta_i - \theta) - D_B\dot\epsilon$ (Ref. 1).

Since the gimbal joint only transmits forces directly, the generalized forces applied by the wheel torquer are written immediately from the concept of virtual work as $Q_{WW} = T_W$ and $Q_{WB} = Q_{Wf_i} = Q_{W\theta} = 0$. The inertia wheel controller commands a wheel speed that is proportional to the angle between the boom's base and the local vertical.[1] Thus, the wheel torquer is represented by $T_W = K_W(\theta_i + \epsilon - D_W\omega_W)$.

As shown in Fig. 2, the orbital station-keeping jet is located at the composite mass center (with the boom vertical and relaxed) and thrust normal to the local vertical. The virtual work done by this jet is

$$
\delta W_J = \mathbf{F}_J{}^T\delta[\mathbf{Z} + (\mathbf{R} - \mathbf{R}_0)] \tag{6}
$$

Since $\mathbf{Z}$ is independent of the generalized coordinates, the virtual displacement can be written as the first order Taylor series $\Sigma[\partial(\mathbf{R} - \mathbf{R}_0)/\partial q_n]\delta q_n$. Substituting this series into Eq. (6) yields

$$
Q_{J\theta} = F_J[M_2L\beta_s\theta_s + M_2L(1 - \beta_c)\theta_c] \tag{7a}
$$

$$
Q_{JB} = -F_JM_2L(\beta + \theta)_c \tag{7b}
$$

$$
Q_{Jf_i} = F_J[M_3\theta_c - M_3(\beta - \theta)_c] \tag{7c}
$$

Solar radiation pressure disturbances are adequately treated as constants over the frequency range of interest here.

Analytical Summary

The nonlinear dynamic equations describing the coupled planar motion of the various structural and control elements are obtained by applying Lagrange's equation to the energy and generalized force expressions developed earlier. The resultant second-order differential equations can be written in the form

$$[\mathbf{m}]\ddot{\mathbf{q}} + [\mathbf{d}]\mathbf{q} + [\mathbf{k}]\mathbf{q} + \mathbf{N} = \mathbf{Q} \tag{8}$$

[**m**], [**d**], and [**k**] are the generalized mass, damping, and spring matrices, respectively. **N** is a column vector of nonlinear kinematic reaction terms, and **Q** is a column vector of generalized forces that are not linear in the state. These elements are partially tabulated in the Appendix. Because of the similarity of the modal dynamics, only the first two have been written explicitly.

Except for the inertia wheel, which appears as a simple reaction element coupled only to the main vehicle, the [**m**] matrix is heavily encumbered. Because of the arbitrary location of the boom's pivot joint, the [**m**] matrix is also a function of time. Although much of the complication in [**m**] arises from the inertial loads imposed by the tip mass and main vehicle on the uniform boom's modal dynamics, relative motion between the mass centers of the various structural members introduces additional dynamic coupling. Since inversion of such a matrix poses no problem in either analog or hybrid computation, in these cases, this approach requires a minimum of coordinates and effort in choosing modal functions that satisfy the prescribed boundary conditions. Admittedly, repeated inversion of an encumbered and time-varying [**m**] is time consuming in digital computations; however, there is no approach to the general nonlinear problem that assures a constant and diagonal [**m**]. When the structural configuration and its attitude are both slowly changing with time, the introduction of synthetic modes as proposed in Ref. 12 enables [**m**] to be reduced to a constant diagonal form. This simplification is achieved at the expense of an increase in problem dimensions and in [**k**] matrix density, so that whether or not computational efficiency is actually improved depends on the application.

An interesting result is that the [**d**] and [**k**] matrices are also nondiagonal and are considerably more dense than their counterparts in classical gravity-gradient systems. This complication arises mainly from application of a torque at the boom's base that obeys a "proportional plus rate" control law. The rate signal here is gimbal angular rate which includes motion induced by the boom's flexure. At first glance, the [**d**] matrix suggests that in addition to controlling the main vehicle's attitude, the gimbal torquer could also provide significant damping of the boom's flexural motion. Actually, time delays and/or quantization errors associated with detecting the gimbal angular rate require great attenu-

Table 1 Nominal system parameters

Uniform boom	
Material density, slugs/in.^{3}	0.9225×10^{-2}
Modulus of elasticity, lb/in.^{2}	1.9×10^7
Diameter, in.	0.5
Material thickness, in.	0.002
Length, ft	150
Tip mass, slugs	0.465
Main vehicle	
Mass, slugs	42.88
Gimbal offset, ft	
g_1	3
g_2	5
Principal inertia	
Roll axis, slug-ft^2	2820
Pitch axis, slug-ft^2	1930

Table 2 Expressions for *M* and *I* parameters

$M_5 = (M_T - M_S M_2^2)L^2/2$	$M_6 = (M_T - M_S M_3^2)/2$
$M_7 = M_S M_2 M_3 L$	$M_8 = M_S M_2 L(1 - M_1)$
$M_9 = M_S M_3(M_1 - 1)$	$M_{10} = M_S M_2 M_3 L^2$
$I_1 = 2M_5 + I_u$	$I_2 = I_F + I_w$
$I_3 = M_8(g_1\beta_S + g_2\beta_c)$	$I_4 = I_1 + I_3$
$I_5 = M_8(g_1\beta_c - g_2\beta_S)$	$I_6 = M_9(g_1\beta_c + g_2\beta_S)$
$I_7 = M_9(g_1\beta_S - g_2\beta_c)$	$I_8 = M_T L - M_7\beta_{2c}$

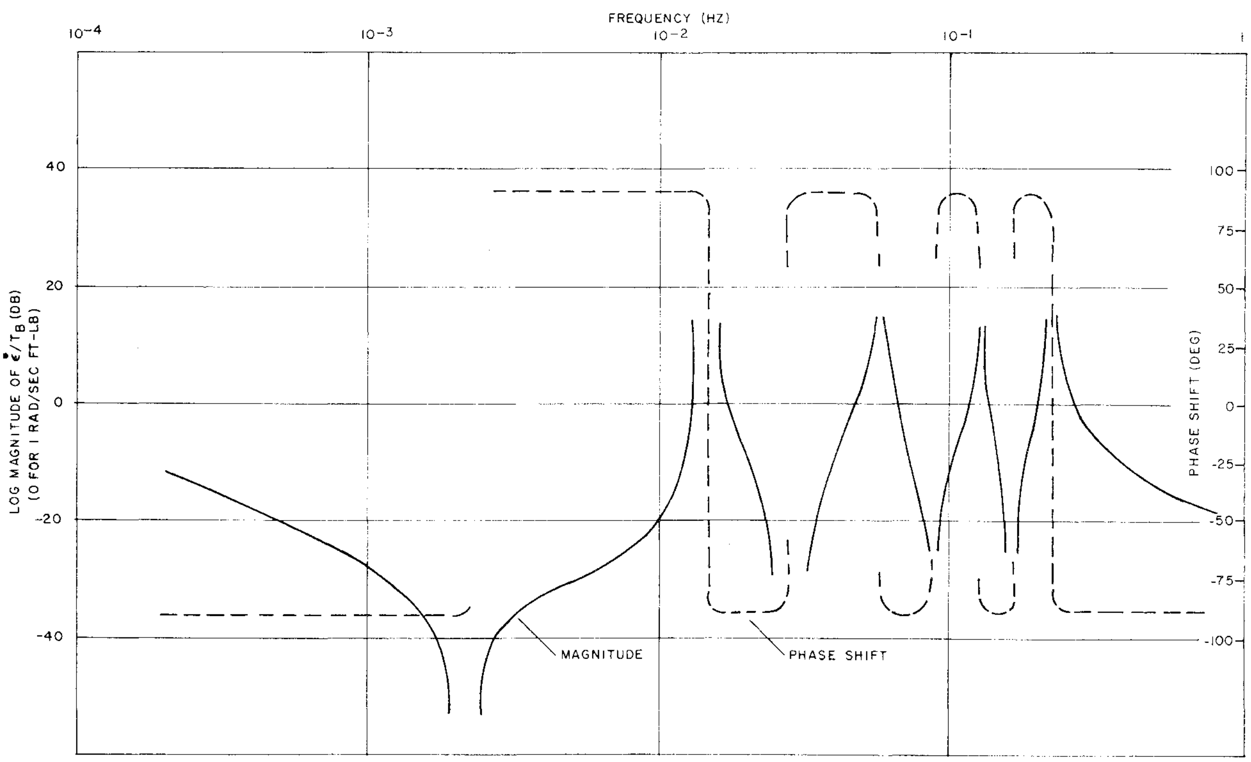

Fig. 3 Frequency response of gimbal rate vs gimbal torque.

ation of loop gain in the vicinity of the boom's inherently lightly damped resonances to avoid inducing excessive flexure. Attempts to raise the boom's resonant frequencies inevitably lead to larger solar radiation pressure disturbances, which degrade steady-state performance. Thus, design of the gimbal torque loop is a classical control systems problem in which transient response is traded against steady-state performance and a rather precise model of the boom's flexural behavior is needed.

Additional complications in the [**d**] matrix stem from the speed-controlled inertia wheel whose control torque is transmitted to the boom through the primary position loop. The inertia wheel actively dampens the gravity-induced librations of the pivoted boom. At frequencies much higher than twice the orbital frequency, the wheel loop gain is rapidly attenuated to avoid degrading the slewing performance of the gimbal torque loop. Thus, the inertia wheel also does not provide significant damping of the boom's flexural motion.

Simulation Results

Equation (8) was programmed on an analog computer utilizing a large communications satellite (see Fig. 1), similar to early versions of the ATS-F and G series as the nominal configuration. Pertinent system parameters are given in Table 1.

Since the characterization of gimbal torquer interaction with the boom's flexural dynamics is an essential element in this model, a rather good test of validity is provided by determining the frequency response of gimbal angular rate versus gimbal torque. With both control loops open, the resultant amplitude and phase response is shown in Fig. 3. For computational efficiency, only the first four flexural modes were included. In the low-frequency region (<0.001 Hz), the amplitude response is attenuated by 6 db/decade while the phase angle remains constant at $-90°$. Of course, these responses typify rigid body motion. At higher frequencies, the pivoted boom exhibits four resonances and four antiresonances. Thus, the effective phase shift is not accumulative. Since the peaks of the amplitude response are not well defined, it is difficult to precisely determine the frequencies at which these resonances and antiresonances occur. As one might expect, however, the resonances occur very near (within machine accuracy) the characteristic frequencies of a classical pin-free boom with tip mass. The antiresonances occur near those frequencies characterizing a classical cantilevered boom with tip mass. Actually, the antiresonances occur at somewhat higher frequencies, since the boom is not cantilevered but is attached to a satellite having finite inertia. By arbitrarily increasing the satellite inertia, agreement within machine accuracy was obtained between the antiresonant frequencies and those exhibited by a cantilevered boom with tip mass.

Appendix: Generalized Dynamical Matrices

When the system dynamics are written in the form of Eq. (8), the generalized coefficient matrices become

$$[\mathbf{m}] = \begin{bmatrix} I_1 & I_4 & (m_1L + I_8) & (m_2L + I_8) & \cdot & \cdot & 0 \\ I_3 & I_2 & I_7 & I_7 & \cdot & \cdot & 0 \\ (M_T + m_1)L & (m_1L + I_7 + I_8) & (2M_6 + M_u/4) & 2M_6 & \cdot & \cdot & 0 \\ (M_T + m_2)L & (m_2L + I_7 + I_8) & 2M_6 & (2M_6 + M_u/4) & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & Iw & 0 & 0 & \cdot & \cdot & Iw \end{bmatrix} \tag{A1}$$

$$[\mathbf{d}] = \begin{bmatrix} D_B & 0 & D_Bf'_1(0) & D_Bf'_2(0) & \cdot & \cdot & 0 \\ -D_B & 0 & -D_Bf'_1(0) & -D_Bf'_2(0) & \cdot & \cdot & -K_wD_w \\ D_Bf'_1(0) & 0 & D_Bf'_1(0)f'_1(0) & D_Bf'_1(0)f'_2(0) & \cdot & \cdot & 0 \\ D_Bf'_2(0) & 0 & D_Bf'_2(0)f'_1(0) & D_Bf'_2(0)f'_2(0) & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & K_wD_w \end{bmatrix} \tag{A2}$$

$$[\mathbf{k}] = \begin{bmatrix} 0 & -K_B & 0 & 0 & \cdot & \cdot & 0 \\ K_w & K_B & K_wf'_1(0) & K_wf'_2(0) & \cdot & \cdot & 0 \\ 0 & -K_Bf'_1(0) & EI\beta_1{}^4L/4 & 0 & \cdot & \cdot & 0 \\ 0 & -K_Bf'_2(0) & 0 & EI\beta_2{}^4L/4 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -K_w & 0 & -K_wf'_1(0) & -K_wf'_2(0) & \cdot & \cdot & 0 \end{bmatrix} \tag{A3}$$

Similarly, those generalized forces that are not linear in the state and all nonlinear kinematic coupling terms are written as

$$\mathbf{Q} = \begin{bmatrix} Q_{JB} - K_B\theta_i \\ (K_B - K_w)\theta_i + F_JM_2L\theta_c \\ -K_Bf'_1(0)\theta_i + Q_{Jf1} \\ -K_Bf'_2(0)\theta_i + Q_{Jf2} \\ \vdots \\ K_w\theta_i \end{bmatrix} \tag{A4}$$

$$\mathbf{N} = \begin{bmatrix} [2M_7\beta_{2S}\dot{\delta} + 4M_6\delta\dot{\delta} + (M_u/2)\sum_i W_i\dot{W}_i](\dot{\beta} + \dot{\theta} + \omega_0) - I_5(\dot{\theta} + \omega_0)^2 \\ I_5(\dot{\beta} + \dot{\theta} + \omega_0)^2 + [(M_u/2)\sum_i W_i\dot{W}_i - 2I_6\dot{\delta}](\dot{\theta} + \omega_0) + 2I_6\dot{\delta}\dot{\beta} \\ [I_5 + M_7\beta_{2c} - M_T\delta + (M_u/4)W_1](\dot{\theta} + \omega_0)^2 \\ [I_6 + M_7\beta_{2c} - M_T\delta + (M_u/4)W_2](\dot{\theta} + \omega_0)^2 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{A5}$$

where the M and I parameters are tabulated in Table 2.

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MARCH 1970 J. SPACECRAFT VOL. 7, NO. 3

A Simple Description of Combined Precession and Nutation in an *N*-Member System of Coaxial, Differentially Spinning Bodies

Donald J. Liska*
University of California, Los Alamos Scientific Laboratory, Los Alamos, N. Mex.

The spin axis motion of an n-body, coaxially-mounted cluster of differentially spinning bodies is analyzed. The description of this motion is made as simple as possible by linearizing the coordinate transformation equations. The resulting motions are then restricted to 10° or so (an acceptable assumption for many applications). The epicycloidal motion is discussed by means of computer plots for different values of momentum, inertia, and torque. Equivalent single bodies with dynamic responses similar to the coaxial system are described. A simple graphical method can be used to plot the spin axis motion for all cases.

Nomenclature

x,y,z = coordinates of body to which torque is applied
ω_x,ω_y = x and y axis angular rates of body of x,y,z coordinates
$\dot{\varphi}$ = spin rate of body with x,y,z coordinates
A = transverse inertia of body with x,y,z coordinates
C = axial inertia of body with x,y,z coordinates
H = angular momentum of body with x,y,z coordinates
A_c = transverse inertia of rest of system
H_c = spin axis angular momentum of rest of system
H_0 = total spin axis angular momentum
T_x = torque applied about x axis of body with x,y,z coordinates
T_{xR},T_{yR} = reaction torques about x and y axes
$\Delta\theta,\Delta\mu$ = spin axis angles of deviation relative to inertial space
t_p = time duration of torque application, measured from zero
t = time variable for all cases
$\dot{\varphi}_e$ = spin rate of equivalent body
R_F = radius of fixed circle in graphical method
R_R = radius of rolling circle in graphical method

Received January 13, 1969; revision received November 10, 1969. The majority of this work was accomplished while the author was employed at The Boeing Company, under sponsorship of the Aerospace Division, Research and Development. This work was also supported in part by the U.S. Atomic Energy Commission.

*Professor.

Introduction

THE usefulness of a first-order linear analysis in understanding the basic characteristics of motion of spinning bodies in space has beeen well demonstrated. In particular, such an analysis can be used to describe the spin-axis motion of a cluster of symmetrical bodies, coaxially mounted, and spin-